$$\lim_{x\to 0}\frac{4x}{\tan x}=\lim_{x\to 0}\frac{4x}{\frac{\sin x}{\cos x}}=\lim_{x\to 0}\frac{4x\cos x}{\sin x}$$

Since $\lim_{x\to 0}\frac{x}{\sin x}=1$ and $\lim_{x\to 0}\cos x=1$, the answer is 4.

Note: Pay careful attention to this next solved problem. It will be very important when you work on problems in Chapter 4.

Problem 7. Find $\lim_{h\to 0}\frac{(5+h)^2-25}{h}$.

Answer: First, expand and simplify the numerator like this:

$$\lim_{h\to 0}\frac{(5+h)^2-25}{h}=\lim_{h\to 0}\frac{25+10h+h^2-25}{h}=\lim_{h\to 0}\frac{10h+h^2}{h}.$$

Next, factor h out of the numerator and the denominator like this:

$$\lim_{h\to 0}\frac{10h+h^2}{h}=\lim_{h\to 0}\frac{h(10+h)}{h}=\lim_{h\to 0}(10+h).$$

Taking the limit you get: $\lim_{h\to 0}(10+h)=10$.

PRACTICE PROBLEM SET 1

Try these 30 problems to test your skill with limits. The answers are in Chapter 21.

1. $\lim_{x\to 8}\left(x^2-5x-11\right)=$

2. $\lim_{x\to 5}\left(\frac{x+3}{x^2-15}\right)=$

3. $\lim_{x\to 0}\pi^2=$

4. $\lim_{x\to 3}\left(\frac{x^2-2x-3}{x-3}\right)=$

5. $\lim_{x\to\infty}\left(\frac{10x^2+25x+1}{x^4-8}\right)=$

6. $\lim_{x\to\infty}\left(\frac{x^4-8}{10x^2+25x+1}\right)=$

7. $\lim_{x\to\infty}\left(\dfrac{x^4-8}{10x^4+25x+1}\right)=$

8. $\lim_{x\to\infty}\left(\dfrac{\sqrt{5x^4+2x}}{x^2}\right)=$

9. $\lim_{x\to6^+}\left(\dfrac{x+2}{x^2-4x-12}\right)=$

10. $\lim_{x\to6^-}\left(\dfrac{x+2}{x^2-4x-12}\right)=$

11. $\lim_{x\to6}\left(\dfrac{x+2}{x^2-4x-12}\right)=$

12. $\lim_{x\to0^+}\left(\dfrac{x}{|x|}\right)=$

13. $\lim_{x\to0^-}\left(\dfrac{x}{|x|}\right)=$

14. $\lim_{x\to7^+}\left(\dfrac{x}{x^2-49}\right)=$

15. $\lim_{x\to7}\left(\dfrac{x}{x^2-49}\right)=$

16. $\lim_{x\to7}\dfrac{x}{(x-7)^2}=$

17. Let $f(x)=\begin{cases}x^2-5, x\le 3\\ x+2, x>3\end{cases}$

Find: (a) $\lim_{x\to3^-} f(x)$; (b) $\lim_{x\to3^+} f(x)$; and (c) $\lim_{x\to3} f(x)$

18. Let $f(x)=\begin{cases}x^2-5, x\le 3\\ x+1, x>3\end{cases}$

Find: (a) $\lim_{x\to3^-} f(x)$; (b) $\lim_{x\to3^+} f(x)$; and (c) $\lim_{x\to3} f(x)$

19. Find $\lim_{x \to \frac{\pi}{4}} 3\cos x$.

20. Find $\lim_{x \to 0} 3\dfrac{x}{\cos x}$.

21. Find $\lim_{x \to 0} 3\dfrac{x}{\sin x}$.

22. Find $\lim_{x \to 0} \dfrac{\sin 3x}{\sin 8x}$.

23. Find $\lim_{x \to 0} \dfrac{\tan 7x}{\sin 5x}$.

24. Find $\lim_{x \to \infty} \sin x$.

25. Find $\lim_{x \to \infty} \sin \dfrac{1}{x}$.

26. Find $\lim_{x \to 0} \dfrac{x^2 \sin x}{1 - \cos^2 x}$.

27. Find $\lim_{x \to 0} \dfrac{\sin^2 7x}{\sin^2 11x}$.

28. Find $\lim_{h \to 0} \dfrac{(3+h)^2 - 9}{h}$.

29. Find $\lim_{h \to 0} \dfrac{\sin(x+h) - \sin x}{h}$.

30. Find $\lim_{h \to 0} \dfrac{\frac{1}{x+h} - \frac{1}{x}}{h}$.

Answer: If you factor the top and bottom of this fraction, you get:

$$f(x)=\frac{2x^2-7x-15}{x^2-x-20}=\frac{(2x+3)(x-5)}{(x+4)(x-5)}$$

Thus, the function has an essential discontinuity at $x = -4$. If we then cancel the term $(x - 5)$, and substitute $x = 5$ into the reduced expression, we get $f(5)=\frac{13}{9}$. Therefore, the function has a removable discontinuity at $\left(5,\frac{13}{9}\right)$.

Note: Don't confuse coordinate parentheses with interval notation. In interval notation, square brackets include endpoints and parentheses do not. For example, the interval $2 \le x \le 4$ is written $[2, 4]$ and the interval $2 < x < 4$ is written $(2, 4)$.

PRACTICE PROBLEM SET 2

Now try these problems. The answers are in Chapter 21.

1. Is the function $f(x)=\begin{cases} x+7, & x<2 \\ 9, & x=2 \\ 3x+3, & x>2 \end{cases}$ continuous at $x = 2$?

2. Is the function $f(x)=\begin{cases} 4x^2-2x, & x<3 \\ 10x-1, & x=3 \\ 30, & x>3 \end{cases}$ continuous at $x = 3$?

3. Is the function $f(x)=\begin{cases} 5x+7, & x<3 \\ 7x+1, & x>3 \end{cases}$ continuous at $x = 3$?

4. Is the function $f(x) = \sec x$ continuous everywhere?

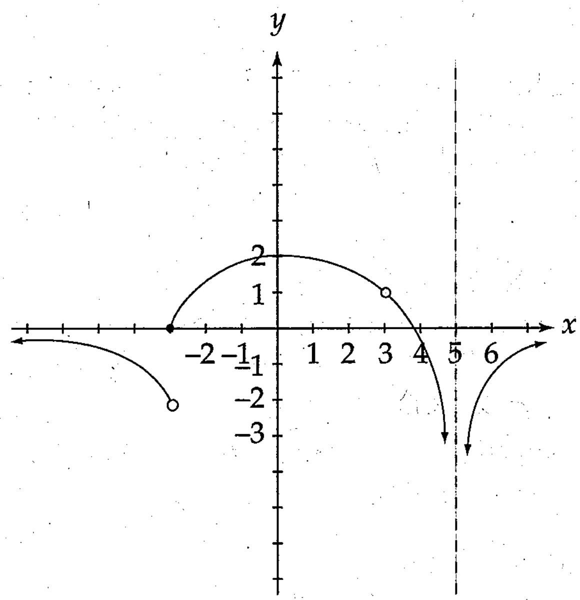
5. Is the function $f(x) = \sec x$ continuous on the interval $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$?

6. Is the function $f(x) = \sec x$ continuous on the interval $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$?

7. For what value(s) of k is the function $f(x) = \begin{cases} 3x^2-11x-4, & x\le 4 \\ kx^2-2x-1, & x>4 \end{cases}$ continuous at $x = 4$?

8. For what value(s) of k is the function $f(x) = \begin{cases} -6x-12, & x<-3 \\ k^2-5k, & x=-3 \\ 6, & x>-3 \end{cases}$ continuous at $x = -3$?

9. At what point is the removable discontinuity for the function $f(x) = \dfrac{x^2+5x-24}{x^2-x-6}$?



10. Given the graph of $f(x)$ above, find:

(a) $\lim\limits_{x \to -\infty} f(x)$

(b) $\lim\limits_{x \to \infty} f(x)$

(c) $\lim\limits_{x \to 3^-} f(x)$

(d) $\lim\limits_{x \to 3^+} f(x)$

(e) $f(3)$

(f) Any discontinuities.

Set 1

1. 13,
2. $\frac{4}{5}$
3. π^2
4. 4
5. ϕ
6. $+\infty$
7. $\frac{1}{10}$
8. $\sqrt{5}$
9. $+\infty$
10. $-\infty$
11. The limit Does Not Exist
12. 1
13. -1
14. ∞
15. DNE
16. ∞
17. (a) 4 (b) 5 (c) DNE
18. (a) 4 (b) 4 (c) 4
19. $\frac{3}{\sqrt{2}}$
20. ϕ
21. 3
22. $\frac{3}{8}$
23. $\frac{7}{5}$
24. DNE
25. 0
26. ϕ
27. $\frac{49}{121}$
28. 6
29. $\cos x$.
30. $-\frac{1}{x^2}$

Set 2

1. Yes, all 3 conditions.

2. No,

3. No.

4. No

5. No

6. Yes

7. continuous for $k = \frac{9}{16}$

8. continuous for $k = 6$ or $k = -1$

9. removable discontinuity at $\left(3, \frac{11}{5}\right)$

10. (a) 0, (b) 0, (c) 1, (d) 1 (e) $f(3)$ DNE

    (f) a jump discontinuity at $x = -3$.

    a removable discontinuity at $x = 3$.

    and an essential discontinuity at $x = 5$

PRACTICE PROBLEM SET 3

Now find the derivative of the following expressions. The answers are in Chapter 21.

1. $f(x) = 5x$ at $x = 3$
2. $f(x) = 4x$ at $x = -8$
3. $f(x) = 2x^2$ at $x = 5$
4. $f(x) = 5x^2$ at $x = -1$
5. $f(x) = 8x^2$
6. $f(x) = -10x^2$
7. $f(x) = 20x^2$ at $x = a$
8. $f(x) = 2x^3$ at $x = -3$
9. $f(x) = -3x^3$
10. $f(x) = x^4$
11. $f(x) = x^5$
12. $f(x) = 2\sqrt{x}$ at $x = 9$
13. $f(x) = 5\sqrt{2x}$ at $x = 8$
14. $f(x) = \sin x$ at $x = \frac{\pi}{3}$
15. $f(x) = \cos x$
16. $f(x) = x^2 + x$
17. $f(x) = x^3 + 3x + 2$
18. $f(x) = \frac{1}{x}$
19. $f(x) = ax^2 + bx + c$
20. $f(x) = \frac{1}{x^2}$

Problem 2. If $y = 9x^4 + 6x^2 - 7x + 11$, then $\frac{dy}{dx} =$

Answer: $\frac{dy}{dx} = 9(4x^3) + 6(2x) - 7(1) + 0 = 36x^3 + 12x - 7$

Problem 3. If $f(x) = 6x^{\frac{3}{2}} - 12\sqrt{x} - \frac{8}{\sqrt{x}} + 24x^{-\frac{3}{2}}$, then $f'(x) =$

Answer: $f'(x) = 6\left(\frac{3}{2}x^{\frac{1}{2}}\right) - \left(\frac{12}{2\sqrt{x}}\right) - 8\left(-\frac{1}{2}x^{-\frac{3}{2}}\right) + 24\left(-\frac{3}{2}x^{-\frac{5}{2}}\right) = 9\sqrt{x} - \frac{6}{\sqrt{x}} + 4x^{-\frac{3}{2}} - 36x^{-\frac{5}{2}}$

How'd you do? Did you notice the changes in notation? How about the fractional powers, radical signs, and x's in denominators? You should be able to switch back and forth between notations, between fractional powers and radical signs, and between negative powers in a numerator and positive powers in a denominator.

PRACTICE PROBLEM SET 4

Find the derivative of each expression and simplify. The answers are in Chapter 21.

1. $(4x^2 + 1)^2$

2. $(x^5 + 3x)^2$

3. $11x^7$

4. $8x^{10}$

5. $18x^3 + 12x + 11$

6. $\frac{1}{2}(x^{12} + 17)$

7. $-\frac{1}{3}(x^9 + 2x^3 - 9)$

8. π^5

9. $\frac{1}{a}\left(\frac{1}{b}x^2 - \frac{2}{a}x - \frac{d}{x}\right)$

10. $-8x^{-8} + 12\sqrt{x}$

11. $6x^{-7} - 4\sqrt{x}$

12. $x^{-5} + \dfrac{1}{x^8}$

13. $\sqrt{x} + \dfrac{1}{x^3}$

14. $(6x^2 + 3)(12x - 4)$

15. $(3 - x - 2x^3)(6 + x^4)$

16. $e^{10} + \pi^3 - 7$

17. $\left(\dfrac{1}{x} + \dfrac{1}{x^2}\right)\left(\dfrac{4}{x^3} - \dfrac{6}{x^4}\right)$

18. $\sqrt{x} + \dfrac{1}{\sqrt{3}}$

19. $(x^2 + 8x - 4)(2x^{-2} + x^{-4})$

20. 0

21. $(x + 1)^3$

22. $\sqrt{x} + \sqrt[3]{x} + \sqrt[3]{x^2}$

23. $x(2x + 7)(x - 2)$

24. $\sqrt{x}\left(\sqrt[3]{x} + \sqrt[5]{x}\right)$

25. $ax^5 + bx^4 + cx^3 + dx^2 + ex + f$

THE PRODUCT RULE

Now that you know how to find derivatives of simple polynomials, it's time to get more complicated. What if you had to find the derivative of this?

$$f(x) = (x^3 + 5x^2 - 4x + 1)(x^5 - 7x^4 + x)$$

You could multiply out the expression and take the derivative of each term, like this:

$$f(x) = x^8 - 2x^7 - 39x^6 + 29x^5 - 6x^4 + 5x^3 - 4x^2 + x$$

And the derivative is:

$$f'(x) = 8x^7 - 14x^6 - 234x^5 + 145x^4 - 24x^3 + 15x^2 - 8x + 1$$

PRACTICE PROBLEM SET 5

Simplify when possible. The answers are in Chapter 21.

1. Find $f'(x)$ if $f(x)=\left(\dfrac{4x^3-3x^2}{5x^7+1}\right)$.

2. Find $f'(x)$ if $f(x)=\left(x^2-4x+3\right)\left(x+1\right)$.

3. Find $f'(x)$ if $f(x)=\left(x+1\right)^{10}$.

4. Find $f'(x)$ if $f(x)=8\sqrt{\left(x^4-4x^2\right)}$.

5. Find $f'(x)$ if $f(x)=\left(\dfrac{x}{x^2+1}\right)^3$.

6. Find $f'(x)$ if $f(x)=\sqrt[4]{\left(\dfrac{2x-5}{5x+2}\right)}$.

7. Find $f'(x)$ if $f(x)=\dfrac{4x^8-\sqrt{x}}{8x^4}$.

8. Find $f'(x)$ if $f(x)=\left(x+\dfrac{1}{x}\right)\left(x^2-\dfrac{1}{x^2}\right)$.

9. Find $f'(x)$ if $f(x)=\left(\dfrac{x}{x+1}\right)^4$.

10. Find $f'(x)$ if $f(x)=\left(x^2+x\right)^{100}$.

11. Find $f'(x)$ if $f(x)=\sqrt{\dfrac{x^2+1}{x^2-1}}$.

12. Find $f'(x)$ at $x = 2$ if $f(x)=\dfrac{(x+4)(x-8)}{(x+6)(x-6)}$.

13. Find $f'(x)$ at $x = 1$ if $f(x)=\dfrac{x^6+4x^3+6}{(x^4-2)^2}$.

14. Find $f'(x)$ at $x = 1$ if $f(x)=\left[\dfrac{x-\sqrt{x}}{x+\sqrt{x}}\right]^2$.

15. Find $f'(x)$ if $f(x)=\dfrac{x^2-3}{(x-3)}$.

16. Find $f'(x)$ at $x = 1$ if $f(x)=(x^4-x^2)(2x^3+x)$.

17. Find $f'(x)$ at $x = 2$ if $f(x)=\dfrac{x^2+2x}{x^4-x^3}$.

18. Find $f'(x)$ if $f(x)=\sqrt{x^4+x^2}$.

19. Find $f'(x)$ at $x = 1$ if $f(x)=\dfrac{x}{(1+x^2)^2}$.

20. Find $\dfrac{dy}{dx}$ if $y=u^2-1$ and $u=\dfrac{1}{x-1}$.

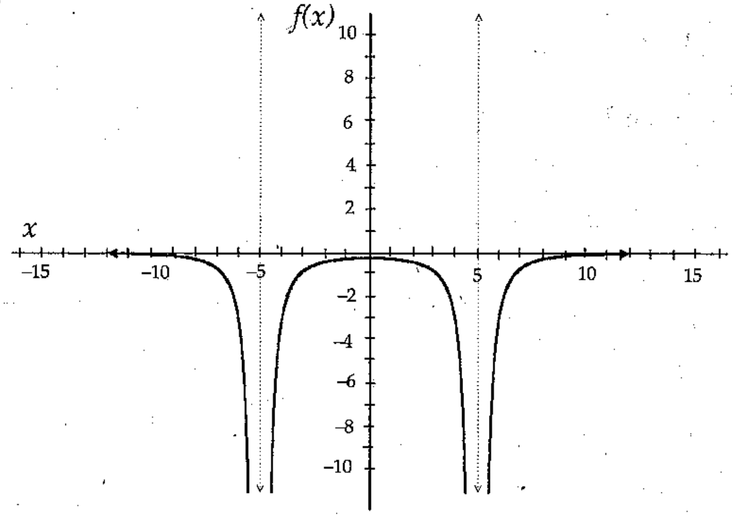
21. Find $\dfrac{dy}{dx}$ at $x = 1$ if $y=\dfrac{t^2+2}{t^2-2}$ and $t=x^3$.

22. Find $\dfrac{dy}{dt}$ if $y=(x^6-6x^5)(5x^2+x)$ and $x=\sqrt{t}$.

23. Find $\dfrac{du}{dv}$ at $v = 2$ if $u=\sqrt{x^3+x^2}$ and $x=\dfrac{1}{v}$.

24. Find $\dfrac{dy}{dx}$ at $x = 1$ if $y=\dfrac{1+u}{1+u^2}$ and $u=x^2-1$.

25. Find $\dfrac{du}{dv}$ if $u = y^3$ and $y=\dfrac{x}{x+8}$ and $x = v^2$.



PRACTICE PROBLEM SET 6

Now try these problems. The answers are in Chapter 21.

1. Find $\frac{dy}{dx}$ if $y = \sin^2 x$.

2. Find $\frac{dy}{dx}$ if $y = \cos x^2$.

3. Find $\frac{dy}{dx}$ if $y = (\tan x)(\sec x)$.

4. Find $\frac{dy}{dx}$ if $y = \cot 4x$.

5. Find $\frac{dy}{dx}$ if $y = \sqrt{\sin 3x}$.

6. Find $\frac{dy}{dx}$ if $y = \frac{1+\sin x}{1-\sin x}$.

7. Find $\frac{dy}{dx}$ if $y = \csc^2 x^2$.

8. Find $\frac{dy}{dx}$ if $y = 2 \sin 3x \cos 4x$.

9. Find $\frac{d^4y}{dx^4}$ if $y = \sin 2x$.

10. Find $\frac{dy}{dx}$ if $y = \sin t - \cos t$ and $t = 1 + \cos^2 x$.

11. Find $\frac{dy}{dx}$ if $y = \left(\frac{\tan x}{1 - \tan x}\right)^2$.

12. Find $\frac{dr}{d\theta}$ if $r = \sec\theta \tan 2\theta$.

13. Find $\frac{dr}{d\theta}$ if $r = \cos(1 + \sin\theta)$.

14. Find $\frac{dr}{d\theta}$ if $r = \frac{\sec\theta}{1 + \tan\theta}$.

15. Find $\frac{dy}{dx}$ if $y = \left(1 + \cot\left(\frac{2}{x}\right)\right)^{-2}$.

16. Find $\frac{dy}{dx}$ if $y = \sin\left(\cos\left(\sqrt{x}\right)\right)$.

SET. 3.

1. 5.
2. 4.
3. 20.
4. -10.
5. $16x$
6. $-20x$
7. $40a$
8. 54
9. $-9x^2$
10. $4x^3$
11. $5x^4$
12. $\frac{1}{3}$
13. $\frac{5}{4}$
14. $\frac{1}{2}$
15. $-\sin x$
16. $2x+1$
17. $3x^2+3$
18. $-\frac{1}{x^2}$
19. $2ax+b$
20. $-\frac{2}{x^3}$

SET 4.

1. $64x^3 + 16x$

2. $10x^9 + 36x^5 + 18x$

3. $77x^6$

4. $80x^9$

5. $54x^2 + 12$

6. $6x^{11}$

7. $-3x^8 - 2x^2$

8. 0

9. $\frac{2}{ab}x - \frac{2}{a^2} + \frac{d}{ax^2}$

10. $64x^{-9} + \frac{6}{\sqrt{x}}$

11. $-42x^{-8} - \frac{2}{\sqrt{x}}$

12. $-\frac{5}{x^6} - \frac{8}{x^9}$

13. $\frac{1}{2\sqrt{x}} - \frac{3}{x^4}$

14. $216x^2 - 48x + 36$

15. $-6 - 36x^2 + 12x^3 - 5x^4 - 14$

16. 0

17. $-\frac{16}{x^5} + \frac{10}{x^6} + \frac{36}{x^7}$

18. $\frac{1}{2\sqrt{x}}$

19. $-\frac{16}{x^2} + \frac{14}{x^3} - \frac{24}{x^4} + \frac{16}{x^5}$

20. 0

21. $3x^2 + 6x + 3$

22. $\frac{1}{2\sqrt{x}} + \frac{1}{3\sqrt[3]{x^2}} + \frac{2}{3\sqrt[3]{x}}$

23. $6x^2 + 6x - 14$

24. $\frac{5}{6\sqrt[6]{x}} + \frac{7}{10\sqrt[10]{x^3}}$

25. $5ax^4 + 4bx^3 + 3cx^2 + 2dx$

SET 5.

1. $\dfrac{-80x^9+75x^8+12x^2-6x}{(5x^7+1)^2}$

2. $3x^2-6x-1$

3. $10(x+1)^9$

4. $\dfrac{16x^2-32}{\sqrt{(x^4-4)}}$

5. $\dfrac{3x^2-3x^4}{(x^2+1)^4}$

6. $\dfrac{1}{4}\left(\dfrac{2x-5}{5x+2}\right)^{-\frac{3}{4}}\left(\dfrac{29}{(5x+2)^2}\right)$

7. $\dfrac{32x^{11}+7x^{\frac{7}{2}}}{16x^8}$

8. $3x^2+1+\dfrac{1}{x^2}+\dfrac{3}{x^4}$

9. $\dfrac{4x^3}{(x+1)^5}$

10. $100(x^2+x)^{99}(2x+1)$

11. $\dfrac{-2x}{(x^2+1)^{\frac{1}{2}}(x^2-1)^{\frac{3}{2}}}$

12. $\dfrac{9}{64}$

13. 106

14. 0

15. $\dfrac{x^2-6x+3}{(x-3)^2}$

16. 6

17. $-\dfrac{7}{4}$

18. $\dfrac{2x^2+1}{\sqrt{x^2+1}}$

19. $-\dfrac{1}{4}$

20. $-\dfrac{2}{(x-1)^3}$

21. -24

22. $\left(t^3-6t^{\frac{5}{2}}\right)\left(5+\dfrac{1}{2\sqrt{t}}\right)$
$+\left(5t+\sqrt{t}\right)\left(3t^2-15t\right.$

23. $-\dfrac{7\sqrt{2}}{16\sqrt{3}}$

24. 2

25. $\dfrac{48v^5}{(v^2+8)^4}$

SET 6.

1. $2\sin x \cdot \cos x = \sin 2x$

2. $-2x \sin(x^2)$

3. $2\sec^3 x - \sec x$

4. $-4\csc^2(4x)$

5. $\dfrac{3\cos 3x}{2\sqrt{\sin 3x}}$

6. $\dfrac{2\cos x}{(1-\sin x)^2}$

7. $-4x \cdot \csc^2(x^2) \cdot \cot(x^2)$

8. $6 \cdot \cos 3x \cdot \cos 4x - 8\sin 3x \cdot \sin 4x$

9. $16 \sin 2x$

10. $[\cos(1+\cos^2 x) + \sin(1+\cos^2 x)](-2\sin x \cdot \cos x)$

11. $\dfrac{2\tan x \cdot \sec^2 x}{(1-\tan x)^3}$

12. $(\sec\theta)(\sec^2(2\theta))(2) + (\tan 2\theta)(\sec\theta \cdot \tan\theta)$

13. $-[\sin(1+\sin\theta)](\cos\theta)$

14. $\dfrac{\sec\theta(\tan\theta - 1)}{(1+\tan\theta)^2}$

15. $\dfrac{-\left(\frac{4}{x^2}\right)\left(\csc^2\left(\frac{2}{x}\right)\right)}{\left(1+\cot\left(\frac{2}{x}\right)\right)^3}$

16. $\cos(\cos\sqrt{x})(-\sin\sqrt{x})($

PRACTICE PROBLEM SET 7

Use implicit differentiation to find the following derivatives. The answers are in Chapter 21.

1. Find $\dfrac{dy}{dx}$ if $x^3 - y^3 = y$.

2. Find $\dfrac{dy}{dx}$ if $x^2 - 16xy + y^2 = 1$.

3. Find $\dfrac{dy}{dx}$ at (2, 1) if $\dfrac{x+y}{x-y} = 3$.

4. Find $\dfrac{dy}{dx}$ if $\cos y - \sin x = \sin y - \cos x$.

5. Find $\dfrac{dy}{dx}$ if $16x^2 - 16xy + y^2 = 1$ at (1, 1).

6. Find $\dfrac{dy}{dx}$ if $x^{\frac{1}{2}} + y^{\frac{1}{2}} = 2y^2$ at (1, 1).

7. Find $\dfrac{dy}{dx}$ if $x \sin y + y \sin x = \dfrac{\pi}{2\sqrt{2}}$ at $\left(\dfrac{\pi}{4}, \dfrac{\pi}{4}\right)$.

8. Find $\dfrac{d^2y}{dx^2}$ if $x^2 + 4y^2 = 1$.

9. Find $\dfrac{d^2y}{dx^2}$ if $\sin x + 1 = \cos y$.

10. Find $\dfrac{d^2y}{dx^2}$ if $x^2 - 4x = 2y - 2$.

SET 7

1. $\dfrac{3x^2}{1+3y^2}$

2. $\dfrac{8y-x}{y-8x}$

3. $\dfrac{1}{2}$

4. $-\dfrac{\sin x+\cos x}{\sin y+\cos y}$

5. $\dfrac{8}{7}$

6. $\dfrac{1}{7}$

7. -1

8. $-\dfrac{1}{16y^3}$

9. $\dfrac{\sin x.\sin^2 y-\cos y.\cos^2 x}{\sin^3 y}$

10. 1

PRACTICE PROBLEM SET 8

Now try these problems. The answers are in Chapter 21.

1. Find the equation of the tangent to the graph of $y = 3x^2 - x$ at $x = 1$.

2. Find the equation of the tangent to the graph of $y = x^3 - 3x$ at $x = 3$.

3. Find the equation of the normal to the graph of $y = \sqrt{8x}$ at $x = 2$.

4. Find the equation of the tangent to the graph of $y = \dfrac{1}{\sqrt{x^2+7}}$ at $x = 3$.

5. Find the equation of the normal to the graph of $y = \dfrac{x+3}{x-3}$ at $x = 4$.

6. Find the equation of the tangent to the graph of $y = 4 - 3x - x^2$ at $(0, 4)$.

7. Find the equation of the tangent to the graph of $y = 2x^3 - 3x^2 - 12x + 20$ at $x = 2$.

8. Find the equation of the tangent to the graph of $y = \dfrac{x^2+4}{x-6}$ at $x = 5$.

9. Find the equation of the tangent to the graph of $y = \sqrt{x^3 - 15}$ at $(4, 7)$.

10. Find the equation of the tangent to the graph of $y = (x^2 + 4x + 4)^2$ at $x = -2$.

11. Find the values of x where the tangent to the graph of $y = 2x^3 - 8x$ has a slope equal to the slope of $y = x$.

12. Find the equation of the normal to the graph of $y = \dfrac{3x+5}{x-1}$ at $x = 3$.

13. Find the values of x where the normal to the graph of $(x-9)^2$ is parallel to the y-axis.

14. Find the coordinates where the tangent to the graph of $y = 8 - 3x - x^2$ is parallel to the x-axis.

15. Find the values of a, b, and c where the curves $y = x^2 + ax + b$ and $y = cx + x^2$ have a common tangent line at $(-1, 0)$.

SET 8

1. $y = 5(x-1)+2$

2. $y - 18 = 24(x-3)$

3. $y - 4 = -(x-2)$

4. $y - \frac{1}{4} = -\frac{3}{64}(x-3)$

5. $y - 7 = \frac{1}{6}(x-4)$

6. $y = -3x+4$

7. $y = 0.$

8. $y + 29 = -39(x-5)$

9. $y - 7 = \frac{24}{7}(x-4)$

10. $y = 0$

11. $x = \pm\sqrt{\frac{3}{2}}$

12. $y - 7 = \frac{1}{2}(x-3)$

13. $x = 9$

14. $\left(-\frac{3}{2}, \frac{41}{4}\right)$

15. $a = 1$
$b = 0$
$c = 1$

PRACTICE PROBLEM SET 9

Now try these problems. The answers are in Chapter 21.

1. Find the values of c that satisfy the MVTD for $f(x) = 3x^2 + 5x - 2$ on the interval $[-1, 1]$.

2. Find the values of c that satisfy the MVTD for $f(x) = x^3 + 24x - 16$ on the interval $[0, 4]$.

3. Find the values of c that satisfy the MVTD for $f(x) = x^3 + 12x^2 + 7x$ on the interval $[-4, 4]$.

4. Find the values of c that satisfy the MVTD for $f(x) = \frac{6}{x} - 3$ on the interval $[1, 2]$.

5. Find the values of c that satisfy the MVTD for $f(x) = \frac{6}{x} - 3$ on the interval $[-1, 2]$.

6. Find the values of c that satisfy Rolle's theorem for $f(x) = x^2 - 8x + 12$ on the interval $[2, 6]$.

7. Find the values of c that satisfy Rolle's theorem for $f(x) = x^3 - x$ on the interval $[-1, 1]$.

8. Find the values of c that satisfy Rolle's theorem for $f(x) = x(1 - x)$ on the interval $[0, 1]$.

9. Find the values of c that satisfy Rolle's theorem for $f(x) = 1 - \frac{1}{x^2}$ on the interval $[-1, 1]$.

10. Find the values of c that satisfy Rolle's theorem for $f(x) = x^{\frac{2}{3}} - x^{\frac{1}{3}}$ on the interval $[0, 1]$.

$\boxed{\text{SET 9}}$

1. $c = 0$

2. $c = \dfrac{4}{\sqrt{3}}$

3. $c = \dfrac{-12 + 8\sqrt{3}}{3} \approx 0.62$

4. $c = \sqrt{2}$

5. No solution

6. $c = 4$

7. $c = \pm \dfrac{1}{\sqrt{3}}$

8. $c = \dfrac{1}{2}$

9. No solution

10. $c = \dfrac{1}{8}$

[illegible].

PRACTICE PROBLEM SET 10

Now try these problems on your own. The answers are in Chapter 21.

1. A rectangle has its base on the x-axis and its two upper corners on the parabola $y = 12 - x^2$. What is the largest possible area of the rectangle?

2. An open rectangular box is to be made from a 9×12 inch piece of tin by cutting squares of side x inches from the corners and folding up the sides. What should x be to maximize the volume of the box?

3. A 384-square-meter plot of land is to be enclosed by a fence and divided into two equal parts by another fence parallel to one pair of sides. What dimensions of the outer rectangle will minimize the amount of fence used?

4. What is the radius of a cylindrical soda can with volume of 512 cubic inches that will use the minimum material?

5. A swimmer is at a point 500 m from the closest point on a straight shoreline. She needs to reach a cottage located 1800 m down shore from the closest point. If she swims at 4 m/s and she walks at 6 m/s, how far from the cottage should she come ashore so as to arrive at the cottage in the shortest time?

6. Find the closest point on the curve $x^2 + y^2 = 1$ to the point $(2, 0)$.

7. A window consists of an open rectangle topped by a semicircle and is to have a perimeter of 288 inches. Find the radius of the semicircle that will maximize the area of the window.

8. The range of a projectile is $R = \dfrac{v_0^{\,2} \sin 2\theta}{g}$, where v_0 is its initial velocity, g is the acceleration due to gravity and is a constant, and θ is its firing angle. Find the angle that maximizes the projectile's range.

9. A computer company determines that its profit equation (in millions of dollars) is given by $P = x^3 - 48x^2 + 720x - 1000$, where x is the number of thousands of units of software sold and $0 \le x \le 40$. Optimize the manufacturer's profit.

$\boxed{\text{SET10}}$

1. 32.

2. $x = \dfrac{7-\sqrt{13}}{2} \doteqdot 1.697$

3. 16 m × 24 m.

4. $R = \sqrt[3]{\dfrac{256}{\pi}}$

5. 1352.786

6. $\left(\dfrac{2}{\sqrt{5}}, \dfrac{1}{\sqrt{5}}\right)$

7. $r = \dfrac{288}{4+\pi}$

8. $\theta = \dfrac{\pi}{4}$

9. \$15 billion

PRACTICE PROBLEM SET 11

It's time for you to try some of these on your own. Sketch each of the graphs below and check the answers in Chapter 21.

1. $y = x^3 - 9x - 6$

2. $y = -x^3 - 6x^2 - 9x - 4$

3. $y = \left(x^2 - 4\right)\left(9 - x^2\right)$

4. $y = \left(x^2 - 4\right)\left(9 - x^2\right)$

5. $y = \dfrac{x^4}{4} - 2x^2$

6. $y = \dfrac{x-3}{x+8}$

7. $y = \dfrac{x^2-4}{x-3}$

8. $y = 3 + x^{\frac{2}{3}}$

9. $y = x^{\frac{2}{3}}\left(3 - 2x^{\frac{1}{3}}\right)$

10. $y = \dfrac{3x^2}{x^2-4}$

SET 11.

1. Min $(\sqrt{3}, -6\sqrt{3}-6)$

Max $(-\sqrt{3}, 6\sqrt{3}-6)$

2. Min $(-3,-4)$

Max $(-1, 0)$

Inflection Point $(-2,-2)$

3. Min $(0, -36)$

Max $\left(\frac{\sqrt{13}}{2}, \frac{25}{4}\right)$ and $\left(-\frac{\sqrt{13}}{2}, \frac{25}{4}\right)$

Infl. P. $\left(\frac{\sqrt{13}}{6}, -\frac{451}{36}\right)$ and $\left(-\frac{\sqrt{13}}{6}, -\frac{451}{36}\right)$

4. Max. $(0,0)$

Min. $(2,-4)$ and $(-2,-4)$

Inf. P. $\left(\frac{2}{\sqrt{3}}, -\frac{20}{9}\right)$ and $\left(-\frac{2}{\sqrt{3}}, -\frac{20}{9}\right)$

5. V.A. $x=-8$.

H.A. $y=1$.

No. Max. Min. Inf. P.

6. V.A. $x=3$

Oblique. A. $y=x+3$

Max. $(3+\sqrt{5}, 6+2\sqrt{5})$

Min. $(3-\sqrt{5}, 6-2\sqrt{5})$

No Inf. P.

7. V.A. $x = 5$, $x = -5$.

H.A. $y = 0$.

No max. min.

Inf. P. $(0,0)$

8. No maxima, minima, Inf. P. Cusp at $(0,3)$

9. Max. $(1,1)$, No Inf. P. Cusp at $(0,0)$

10. Max. $(0,0)$.

V.A. $x = 2$, $x = -2$

H.A. $y = 3$.

No Infl. P.

PRACTICE PROBLEM SET 12

Now try these problems on your own. The answers are in Chapter 21.

1. Oil spilled from a tanker spreads in a circle whose circumference increases at a rate of 40 ft/sec. How fast is the area of the spill increasing when the circumference of the circle is 100π ft?
2. A spherical balloon is inflating at a rate of 27π in^3/sec. How fast is the radius of the balloon increasing when the radius is 3 in?
3. Cars A and B leave a town at the same time. Car A heads due south at a rate of 80 km/hr and car B heads due west at a rate of 60 km/hr. How fast is the distance between the cars increasing after three hours?
4. A cylindrical tank with a radius of 6 meters is filling with fluid at a rate of 108π m^3/sec. How fast is the height increasing?

5. The sides of an equilateral triangle are increasing at the rate of 27 in/sec. How fast is the triangle's area increasing when the sides of the triangle are each 18 inches long?
6. An inverted conical container has a diameter of 42 in and a depth of 15 in. If water is flowing out of the vertex of the container at a rate of 35π in^3/sec, how fast is the depth of the water dropping when the height is 5 inches?
7. A boat is being pulled toward a dock by a rope attached to its bow through a pulley on the dock 7 feet above the bow. If the rope is hauled in at a rate of 4 ft/sec, how fast is the boat approaching the dock when 25 feet of rope is out?
8. A 6-foot-tall woman is walking at the rate of 4 ft/sec away from a street lamp that is 24 feet tall. How fast is the length of her shadow changing?
9. The voltage, V, in an electrical circuit is related to the current, I, and the resistance, R, by the equation $V = IR$. The current is decreasing at -4 amps/sec as the resistance increases at 20 ohms/sec. How fast is the voltage changing when the voltage is 100 volts and the current is 20 amps?
10. The minute hand of a clock is 6 inches long. Starting from noon, how fast is the area of the sector swept out by the minute hand increasing in in^2/min at any instant?

POSITION, VELOCITY, AND ACCELERATION

Almost every AP exam has a question on position, velocity, or acceleration. It's one of the traditional areas of physics where calculus comes in handy. Some of these problems require the use of integral calculus, which we won't talk about until the second half of this book. So, this unit is divided in half; you'll see the other half later.

$\boxed{\text{SET 12.}}$

1. $2000\ ft^2/s$

2. $\frac{3}{4}\ in/s$

3. $100\ km/hr$

4. $3\ m/s$

5. $243\sqrt{3}\ in^2/s$

6. $-\frac{5}{7}\ in/s$

7. $-\frac{25}{6}\ ft/s$

8. $\frac{4}{3}\ ft/s$

9. $380\ volts/s$

10. $\frac{3\pi}{5}\ in^2/min$

PRACTICE PROBLEM SET 13

Now try these problems. The answers are in Chapter 21.

1. Find the velocity and acceleration of a particle whose position function is $x(t) = t^3 - 9t^2 + 24t$, $t > 0$.

2. Find the velocity and acceleration of a particle whose position function is $x(t) = \sin(2t) + \cos(t)$.

3. If the position function of a particle is $x(t) = \frac{t}{t^2+9}$, $t > 0$, find when the particle is changing direction.

4. If the position function of a particle is $x(t) = \sin\left(\frac{t}{2}\right)$, $0 < t < 4\pi$, find when the particle is changing direction.

5. If the position function of a particle is $x(t) = 3t^2 + 2t + 4$, $t > 0$, find the distance that the particle travels from $t = 2$ to $t = 5$.

6. If the position function of a particle is $x(t) = t^2 + 8t$, $t > 0$, find the distance that the particle travels from $t = 0$ to $t = 4$.

7. If the position function of a particle is $x(t) = 2\sin^2 t + 2\cos^2 t$, $t > 0$, find the velocity and acceleration of the particle.

8. If the position function of a particle is $x(t) = t^3 + 8t^2 - 2t + 4$, $t > 0$, find when the particle is changing direction.

9. If the position function of a particle is $x(t) = 2t^3 - 6t^2 + 12t - 18$, $t > 0$, find when the particle is changing direction.

10. If the position function of a particle is $x(t) = \sin^2 2t$, $t > 0$, find the distance that the particle travels from $t = 0$ to $t = 2$.

SET 13

1. $v(t) = 3t^2 - 18t + 24$

   $a(t) = 6t - 18.$

2. $v(t) = 2\cos(2t) - \sin(t)$

   $a(t) = -4\sin(2t) - \cos(t)$

3. $t = 3$

4. $t = \pi$ and $t = 3\pi$

5. distance $= 69$

6. 48.

7. Velocity $= 0$

   acceleration $= 0$

8. $t = \dfrac{-8 + \sqrt{70}}{3}$

9. X.

10. $2 + \sin^2 K$

    ≈ 2.573

PRACTICE PROBLEM SET 14

Now find the derivative of each of the following functions. The answers are in Chapter 21.

1. $f(x) = \ln\left(x^4 + 8\right)$
2. $f(x) = \ln\left(3x\sqrt{3+x}\right)$
3. $f(x) = \ln\left(\cot x - \csc x\right)$
4. $f(x) = x \ln \cos 3x - x^3$
5. $f(x) = \ln\left(\dfrac{5x^2}{\sqrt{5+x^2}}\right)$
6. $f(x) = e^{x\cos x}$
7. $f(x) = e^{-3x} \sin 5x$
8. $f(x) = \dfrac{e^{\tan 4x}}{4x}$
9. $f(x) = e^{\pi x} - \ln e^{\pi x}$
10. $f(x) = \log_{12}\left(x^3\right)$
11. $f(x) = \log_6\left(3x \tan x\right)$
12. $f(x) = \dfrac{\log_4 x}{e^{4x}}$
13. $f(x) = \log\sqrt{10^{3x}}$
14. $f(x) = \ln x \log x$
15. $f(x) = e^{3x} - 3^{ex}$
16. $f(x) = 10^{\sin x}$
17. $f(x) = 5^{\tan x}$
18. $f(x) = \ln\left(10^x\right)$
19. $f(x) = x^5 5^x$

SET 14

1. $f'(x) = \dfrac{4x^3}{x^4+8}$

2. $f'(x) = \dfrac{1}{x} + \dfrac{1}{6+2x}$

3. $f'(x) = \csc x$

4. $f'(x) = \ln(\cos 3x) - 3x \tan 3x - 3x^2$

5. $f'(x) = \dfrac{2}{x} - \dfrac{x}{5+x^2}$

6. $f'(x) = e^{x\cos x}(\cos x - x \cdot \sin x)$

7. $f'(x) = -3e^{-3x}\sin 5x + 5 \cdot e^{-3x}\cos 5x$

8. $f'(x) = e^{\tan 4x} \dfrac{4x \cdot \sec^2 4x - 1}{4x^2}$

9. $f'(x) = \pi e^{\pi x} - \pi$

10. $f'(x) = \dfrac{3}{x \cdot \ln 12}$

11. $f'(x) = \dfrac{1}{\ln 6}\left(\dfrac{1}{x} + \dfrac{\sec^2 x}{\tan x}\right)$

12. $f'(x) = \dfrac{1}{x \cdot e^{4x}\ln 4} - \dfrac{4\log_4 x}{e^{4x}}$

13. $f'(x) = \frac{3}{2}$

14. $f'(x) = \frac{2 \ln x}{x \cdot \ln 10}$

15. $f'(x) = 3e^{3x} - 3^{ex}(e) \cdot (\ln 3)$

16. $f'(x) = 10^{\sin x}(\cos x)(\ln 10)$

17. $f'(x) = 5^{\tan x} \cdot (\sec^2 x) \cdot \ln 5$

18. $f'(x) = \ln 10$

19. $f'(x) = x^4 5^x (5 + x \ln 5)$

$$\frac{dy}{dx} = 2 - 3x^2$$

Next, find the value of x where $y = 1$. By inspection, $y = 1$ when $x = 1$.
Then, we use the formula to find the derivative of the inverse:

$$\frac{1}{\left.\frac{dy}{dx}\right|_{x=1}} = \frac{1}{\left.\left(2 - 3x^2\right)\right|_{x=1}} = -1$$

PRACTICE PROBLEM SET 15

Find a derivative of the inverse of each of the following functions. The answers are in Chapter 21.

1. $y = x + \frac{1}{x}$ at $y = \frac{17}{4}$; where $x>1$
2. $y = 3x - 5x^3$ at $y = 2$
3. $y = e^x$ at $y = e$
4. $f(x) = x^7 - 2x^5 + 2x^3$ at $f(x) = 1$

5. $y = x + x^3$ at $y = -2$
6. $y = 4x - x^3$ at $y = 3$
7. $y = \ln x$ at $y = 0$
8. $y = x^{\frac{1}{3}} + x^{\frac{1}{5}}$ at $y = 2$

$\boxed{\text{SET 15}}$

1. $\frac{16}{15}$

2. $-\frac{1}{12}$

3. $\frac{1}{e}$

4. $\frac{1}{3}$

5. $\frac{1}{4}$

6. 1

7. 1

8. $\frac{15}{8}$

Problem 3. A particle's position at time t is determined by the equations $x = 3 + 2t^2$ and $y = 4t^4$, $t \geq 0$. Find the x- and y-components of the particle's velocity and the times when these components are equal.

Answer: First, figure out the x- and y-components of the velocity:

$$\frac{dx}{dt} = 4t \text{ and } \frac{dy}{dt} = 16t^3$$

These are equal when $16t^3 = 4t$. Solving for t:

$$t = 0, \pm\frac{1}{2}$$

Throw out the negative value of t, the answer is $t = 0, \frac{1}{2}$.

PRACTICE PROBLEM SET 16

Now try these problems. The answers are in Chapter 21.

1. Find the Cartesian equation of the curve represented by $x = \sec^2 t - 1$ and $y = \tan t$, $-\frac{\pi}{2} < t < \frac{\pi}{2}$.

2. Find the Cartesian equation of the curve represented by $x = t$ and $y = \sqrt{1-t^2}$, $-1 < t < 1$.

3. Find the Cartesian equation of the curve represented by $x = 4t + 3$ and $y = 16t^2 - 9$, $-\infty < t < \infty$.

4. Find the equation of the tangent line to $x = t^2 + 4$ and $y = 8t$ at $t = 6$.

5. Find the equation of the tangent line to $x = \sec t$ and $y = \tan t$ at $t = \frac{\pi}{4}$.

6. The motion of a particle is given by $x = -2t^2$ and $y = t^3 - 3t + 9$, $t \geq 0$. Find the coordinates of the particle when its instantaneous direction of motion is horizontal.

7. The motion of a particle is given by $x = \ln t$ and $y = t^2 - 4t$. Find the coordinates of the particle when its instantaneous direction of motion is horizontal.

8. The motion of a particle is given by $x = 2\sin t - 1$ and $y = \sin t - \frac{t}{2}$, $0 \leq t < 2\pi$. Find the times when the horizontal and vertical components of the particle's velocity are the same.

SET 16.

1. $x = y^2$

2. $y = \sqrt{1-x^2}$

3. $y = x^2 - 6x$

4. $y - 48 = \frac{2}{3}(x-40)$

5. $y - 1 = \sqrt{2}(x - \sqrt{2})$

6. $(-2, 7)$

7. $(\ln 2, -4)$

8. $t = \frac{2\pi}{3}, \frac{4\pi}{3}$

PRACTICE PROBLEM SET 17

Now find these limits using L'Hôpital's Rule. The answers are in Chapter 21.

1. Find $\lim\limits_{x\to 0}\dfrac{\sin 3x}{\sin 4x}$.

2. Find $\lim\limits_{x\to \pi}\dfrac{x-\pi}{\sin x}$.

3. Find $\lim\limits_{x\to 0}\dfrac{x-\sin x}{x^3}$.

4. Find $\lim\limits_{x\to 0}\dfrac{e^{3x}-e^{5x}}{x}$.

5. Find $\lim\limits_{x\to 0}\dfrac{\tan x-x}{\sin x-x}$.

6. Find $\lim\limits_{x\to \infty}\dfrac{x^5}{e^{5x}}$.

7. Find $\lim\limits_{x\to \infty}\dfrac{x^5+4x^3-8}{7x^5-3x^2-1}$.

8. Find $\lim\limits_{x\to 0^+}\dfrac{\ln(\sin x)}{\ln(\tan x)}$.

9. Find $\lim\limits_{x\to 0^+}\dfrac{\cot 2x}{\cot x}$.

10. Find $\lim\limits_{x\to 0^+}\dfrac{x}{\ln(x+1)}$.

DIFFERENTIALS

Sometimes this is called "linearization." A differential is a very small quantity that corresponds to a change in a number. We use the symbol Δx to denote a differential. What are differentials used for? The AP exam mostly wants you to use them to approximate the value of a function or to find the error of an approximation.

Recall the formula for the definition of the derivative:

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

SET 17.

1. $\frac{3}{4}$

2. 1

3. $\frac{1}{6}$

4. -2

5. -2

6. 0

7. $\frac{1}{7}$

8. 1

9. $\frac{1}{2}$

10. 1.

$$S = 4\pi r^2$$

The formula says that $dS = S'dr$, so first, we find the derivative of the surface area ($S' = 8\pi r$) and plug away:

$$dS = 8\pi r dr = 8\pi(4)(\pm.01) = \pm 1.0053$$

This looks like a big error, but given that the surface area of a sphere with radius 4 is approximately 201 cm^2, the error is quite small.

PRACTICE PROBLEM SET 18

Use the differential formulas in this chapter to solve these problems. The answers are in Chapter 21.

1. Approximate $\sqrt{25.02}$.
2. Approximate $\sqrt[3]{63.97}$.
3. Approximate tan 61°.
4. Approximate $(9.99)^3$.
5. The side of a cube is measured to be 6 in. with an error of ± 0.02 in. Estimate the error in the volume of the cube.
6. When a spherical ball bearing is heated, its radius increases by 0.01 mm. Estimate the change in volume of the ball bearing when the radius is 5 mm.
7. A side of an equilateral triangle is measured to be 10 cm. Estimate the change in the area of the triangle when the side shrinks to 9.8 cm.
8. A cylindrical tank is constructed to have a diameter of 5 meters and a height of 20 meters. Find the error in the volume if:

   (a) the diameter is exact, but the height is 20.1 meters; and

   (b) the height is exact, but the diameter is 5.1 meters.

LOGARITHMIC DIFFERENTIATION

There's one last topic in differential calculus that you BC students need to know: logarithmic differentiation. It's a very simple and handy technique used to find the derivatives of expressions that involve a lot of algebra. By employing the rules of logarithms, we can find the derivatives of expressions that would otherwise require a messy combination of the Chain Rule, the Product Rule, and the Quotient Rule.

First, let's review a couple of rules of logarithms (remember, when we refer to a logarithm in calculus, we mean the natural log (base e), not the common log):

$$\ln A + \ln B = \ln(AB)$$
$$\ln A - \ln B = \ln\left(\frac{A}{B}\right)$$
$$\ln A^B = B \ln A$$

For example, we can rewrite $\ln(x+3)^5$ as $5\ln(x+3)$.

As a quick review exercise, how could you rewrite:

$$\ln \frac{x^2}{(\ldots)^2}$$

SET 18.

1. 5.002
2. 3.999375
3. 1.802
4. 997
5. ± 2.16 in^3
6. $\pi = 3.142$ mm^3
7. -1.732 cm^2
8. (a) 1.963 m^3
   (b) 15.708 m^3

Take the derivative of both sides:

$$\frac{1}{y}\frac{dy}{dx}=\frac{2x-5}{x^2-5x}-\frac{2\sin x}{\cos x}-\frac{15x^2}{x^3+1}$$

$$\frac{1}{y}\frac{dy}{dx}=\frac{2x-5}{x^2-5x}-2\tan x-\frac{15x^2}{x^3+1}$$

And multiply by y:

$$\frac{dy}{dx}=y\left[\frac{2x-5}{x^2-5x}-2\tan x-\frac{15x^2}{x^3+1}\right]$$

One more time.

PRACTICE PROBLEM SET 19

Use logarithmic differentiation to find the derivative of each of the following problems. The answers are in Chapter 21.

1. $y=x\left(\sqrt[4]{1-x^3}\right)$

2. $y=\sqrt{\dfrac{1+x}{1-x}}$

3. $y=\dfrac{\left(x^3+5\right)^{\frac{3}{2}}\sqrt[3]{4-x^2}}{x^4-x^2+6}$

4. $y=\dfrac{\sin x\ \cos x}{\sqrt{x^3-4}}$

5. $y=\dfrac{\left(4x^2-8x\right)^3\left(5-3x^4+7x\right)^4}{\left(x^2+x\right)^3}$

6. $y=\dfrac{x-1}{x\tan x}$

7. $y=\left(x-x^2\right)^2\left(x^3+x^4\right)^3\left(x^6-x^5\right)^4$

8. $y=\sqrt[4]{\dfrac{x(1-x)(1+x)}{\left(x^2-1\right)(5-x)}}$

SET 19

1. $y\left[\dfrac{1}{x} - \dfrac{3x^2}{4(1-x^3)}\right]$

2. $\dfrac{y}{2}\left[\dfrac{1}{1+x} + \dfrac{1}{1-x}\right]$

3. $y\left\{\dfrac{9x^2}{2(x^3+5)} - \dfrac{2x}{3(4-x^2)} - \dfrac{4x^3-2x}{(x^4-x^2+6)}\right\}$

4. $y\left\{\cot x - \tan x - \dfrac{3x^2}{2(x^3-4)}\right\}.$

5. $y\left\{\dfrac{6(x-1)}{(x^2-2x)} + \dfrac{4(-12x^3+7)}{(5-3x^4+7x)} - \dfrac{3(2x+1)}{(x^2+x)}\right\}$

6. $y\left\{\dfrac{1}{x-1} - \dfrac{1}{x} - \sec x \;\; \csc x\right\}.$

7. $y\left\{\dfrac{2(1-2x)}{(x-x^2)} + \dfrac{3(3x^2+4x^3)}{(x^3+x^4)} + \dfrac{4(6x^5-5x^4)}{(x^6-x^5)}\right\}$

8. $\dfrac{y}{4}\left\{\dfrac{1}{x} - \dfrac{1}{1-x} + \dfrac{1}{1+x} - \dfrac{2x}{x^2-1} + \dfrac{1}{5-x}\right\}$

PRACTICE PROBLEM SET 20

Now evaluate the following integrals. The answers are in Chapter 21.

1. $\int \frac{1}{x^4} dx$

2. $\int \frac{5}{\sqrt{x}} dx$

3. $\int \frac{x^5+7}{x^2} dx$

4. $\int \left(5x^4-3x^2+2x+6\right) dx$

5. $\int \left(3x^{-3}-2x^{-2}+x^4+16x^7\right) dx$

6. $\int \left(1+x^2\right)\left(x-2\right) dx$

7. $\int x^{\frac{1}{3}}\left(2+x\right) dx$

8. $\int \left(x^3+x\right)^2 dx$

9. $\int \frac{x^6-2x^4+1}{x^2} dx$

10. $\int x\left(x-1\right)^3 dx$

11. $\int \left(\cos x-5\sin x\right) dx$

12. $\int \sec x\left(\sec x+\tan x\right) dx$

13. $\int \left(\sec^2 x+x\right) dx$

14. $\int \frac{\sin x}{\cos^2 x} dx$

15. $\int \frac{\cos^3 x+4}{\cos^2 x} dx$

16. $\int \frac{\sin 2x}{\cos x} dx$

17. $\int \left(1+\cos^2 x \sec x\right) dx$

18. $\int \left(\tan^2 x\right) dx$

19. $\int \frac{1}{\csc x} dx$

20. $\int \left(x - \frac{2}{\cos^2 x}\right) dx$

U-SUBSTITUTION

When we discussed differentiation, one of the most important techniques we mastered was the Chain Rule. Now, you'll learn the integration corollary of the Chain Rule (called *u*-substitution), which we use when the integrand is a composite function. All you do is replace the function with *u*, and then you can integrate the simpler function using the Power Rule (as shown below):

$$\int u^n du = \frac{u^{n+1}}{n+1} + C$$

Suppose you have to integrate $\int (x-4)^{10} dx$. You could expand out this function and integrate each term, but that'll take a while. Instead, you can follow these four steps:

Step 1: Let $u = x - 4$. Then $\frac{du}{dx} = 1$ (rearrange this to get $du = dx$).

Step 2: Substitute $u = x - 4$ and $du = dx$ into the integrand:

$$\int u^{10} du$$

Step 3: Integrate:

$$\int u^{10} du = \frac{u^{11}}{11} + C$$

Step 4: Substitute back for *u*:

$$\frac{(x-4)^{11}}{11} + C$$

SET 20

1. $-\frac{1}{3x^3} + C$

2. $10\sqrt{x} + C$

3. $\frac{x^4}{4} - \frac{7}{x} + C$

4. $x^5 - x^3 + x^2 + 6x + C$

5. $-\frac{3}{2x^2} + \frac{2}{x} + \frac{x^5}{5} + 2x^8 + C$

6. $\frac{x^4}{4} - \frac{2x^3}{3} + \frac{x^2}{2} - 2x + C$

7. $\frac{3x^{\frac{4}{3}}}{2} + \frac{3x^{\frac{7}{3}}}{7} + C$

8. $\frac{x^7}{7} + \frac{2x^5}{5} + \frac{x^3}{3} + C$

9. $\frac{x^5}{5} - \frac{2x^3}{3} - \frac{1}{x} + C$

10. $\frac{x^5}{5} - \frac{3x^4}{4} + x^3 - \frac{x^2}{2} + C$

11. $\sin x + 5\cos x + C$

12. $\tan x + \sec x + C$

13. $\tan x + \frac{x^2}{2} + C$

14. $\sec x + C$

15. $\sin x + 4\tan x + C$

16. $-2\cos x + C$

17. $x + \sin x + C$

18. $\tan x - x + C$

19. $-\cos x + C$

20. $\frac{x^2}{2} - 2\tan x + C$

Problem 4. Evaluate $\int \tan\frac{x}{3}\ \sec^2\frac{x}{3}\,dx$.

Answer: Let $u = \tan\frac{x}{3}$ and $du = \frac{1}{3}\sec^2\frac{x}{3}\,dx$. Then $3du = \sec^2\frac{x}{3}\,dx$.

Substituting, we get:

$$3\int u\,du = \frac{3}{2}u^2 + C$$

Then substitute back:

$$\frac{3}{2}\tan^2\frac{x}{3} + C$$

PRACTICE PROBLEM SET 21

Now evaluate the following integrals. The answers are in Chapter 21.

1. $\int \sin 2x \cos 2x\,dx$

2. $\int \frac{3x\,dx}{\sqrt[3]{10-x^2}}$

3. $\int x^3\sqrt{5x^4+20}\,dx$

4. $\int \frac{dx}{(x-1)^2}$

5. $\int (x^2+1)(x^3+3x)^{-5}\,dx$

6. $\int \frac{1}{\sqrt{x}}\sin\sqrt{x}\,dx$

7. $\int x^2\sec^2 x^3\,dx$

8. $\int \frac{\cos\left(\frac{3}{x}\right)}{x^2}dx$

9. $\int \frac{\sin 2x}{(1-\cos 2x)^3}dx$

10. $\int \sin(\sin x)\cos x\,dx$

SET 21

1. $\dfrac{\sin^2 2x}{4} + C$

2. $-\frac{9}{4}(10-x^2)^{\frac{2}{3}} + C$

3. $\frac{1}{30}(5x^4+20)^{\frac{3}{2}} + C$

4. $-\dfrac{1}{x-1} + C$

5. $-\dfrac{1}{12(x^3+3x)^4} + C$

6. $-2\cos\sqrt{x} + C$

7. $\frac{1}{3}\tan(x^3) + C$

8. $-\frac{1}{3}\sin\left(\frac{3}{x}\right) + C$

9. $-\frac{1}{4}(1-\cos 2x)^{-2} + C$

10. $-\cos(\sin x) + C$

PRACTICE PROBLEM SET 22

Here's a great opportunity to practice finding the area beneath a curve and evaluating integrals. The answers are in Chapter 21.

1. Find the area under the curve $y = 2x - x^2$ from $x = 1$ to $x = 2$ with $n = 4$ left-endpoint rectangles.

2. Find the area under the curve $y = 2x - x^2$ from $x = 1$ to $x = 2$ with $n = 4$ right-endpoint rectangles.

3. Find the area under the curve $y = 2x - x^2$ from $x = 1$ to $x = 2$ using the Trapezoid Rule with $n = 4$.

4. Find the area under the curve $y = 2x - x^2$ from $x = 1$ to $x = 2$ using the Midpoint Formula with $n = 4$.

5. Find the area under the curve $y = 2x - x^2$ from $x = 1$ to $x = 2$.

6. Evaluate $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x \, dx$.

7. Evaluate $\int_{1}^{9} 2x\sqrt{x} \, dx$.

8. Evaluate $\int_{0}^{1} \left(x^4 - 5x^3 + 3x^2 - 4x - 6\right) dx$.

9. Evaluate $\int_{-4}^{4} |x| \, dx$.

10. Evaluate $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin x \, dx$.

11. Suppose we are given the following table of values for x and $g(x)$:

x	0	1	3	5	9	14
$g(x)$	10	8	11	17	20	23

Use a left-hand Riemann sum with 5 subintervals indicated by the data in the table to approximate $\int_{0}^{14} g(x)dx$.

SET22.

1. $\frac{25}{32}$

2. $\frac{17}{32}$

3. $\frac{21}{32}$

4. $\frac{43}{64}$

5. $\frac{2}{3}$

6. 2.

7. $\frac{968}{5}$

8. $-\frac{161}{20}$

9. 16

10. $\emptyset$

11. 216

As in the previous example, let's make a table of values of the accumulation function for different values of x:

x	1	2	3	4
$F(x)$	$\frac{1}{3}$	$\frac{8}{3}$	9	$\frac{64}{3}$

We can see that the values of $F(x)$ will increase as x increases.

PRACTICE PROBLEM SET 23

Now try these problems. The answers are in Chapter 21.

1. Find the average value of $f(x) = 4x \cos x^2$ on the interval $\left[0, \sqrt{\frac{\pi}{2}}\right]$.

2. Find the average value of $f(x) = \sqrt{x}$ on the interval [0, 16].

3. Find the average value of $f(x) = \sqrt{1-x}$ on the interval [–1, 1].

4. Find the average value of $f(x) = 2|x|$ on the interval [–1, 1].

5. Find $\frac{d}{dx}\int_1^x \sin^2 t\, dt$.

6. Find $\frac{d}{dx}\int_5^{3x} \left(t^2 - t\right)dt$.

7. Find $\frac{d}{dx}\int_0^{x^2} |t|\, dt$.

8. Find $\frac{d}{dx}\int_1^{x} -2\cos t\, dt$.

$\boxed{\text{SET 23}}$

1. $2\sqrt{\dfrac{2}{\pi}}$

2. $\dfrac{8}{3}$

3. $\dfrac{2\sqrt{2}}{3}$

4. 1

5. $\sin^2 x$

6. $27x^2-9x$

7. $2x^3$

8. $-2\cos x$.

Now get back to the expression as a function of x:

$$\frac{1}{3\ln 2}2^{3x}+C=\frac{1}{\ln 8}2^{3x}+C$$

PRACTICE PROBLEM SET 24

Evaluate the following integrals. The answers are in Chapter 21.

1. $\int \frac{\sec^2 x}{\tan x}dx$

2. $\int \frac{\cos x}{1-\sin x}dx$

3. $\int \frac{1}{x\ln x}dx$

4. $\int \frac{1}{x}\cos(\ln x)dx$

5. $\int \frac{\sin x-\cos x}{\cos x}dx$

6. $\int \frac{dx}{\sqrt{x}\left(1+2\sqrt{x}\right)}$

7. $\int \frac{e^x dx}{1+e^x}$

8. $\int xe^{5x^2-1}dx$

9. $\int e^x\cos\left(2+e^x\right)dx$

10. $\int \frac{e^x+e^{-x}}{e^x-e^{-x}}dx$

11. $\int x4^{-x^2}dx$

12. $\int 7^{\sin x}\cos x\,dx$

SET24.

1. $\ln|\tan x| + C$

2. $-\ln|1-\sin x| + C$

3. $\ln|\ln x| + C$

4. $\sin(\ln x) + C$

5. $-\ln|\cos x| - x + C$

6. $\ln(1+2\sqrt{x}) + C$

7. $\ln(1+e^x) + C$

8. $\frac{1}{10}e^{5x^2} + C$

9. $\sin(2+e^x) + C$

10. $\ln|e^x - e^{-x}| + C$

11. $-\frac{4^{-x^2}}{\ln 16} + C$

12. $\frac{7^{\sin x}}{\ln 7} + C$

Next, find where the two curves intersect. By setting $y^3 - y = 0$, you'll find that they intersect at $y = -1$, $y = 0$, and $y = 1$. Notice that the curve is to the right of the y-axis from $y = -1$ to $y = 0$ and to the left of the y-axis from $y = 0$ to $y = 1$. Thus, the region must be divided into two parts: from $y = -1$ to $y = 0$ and from $y = 0$ to $y = 1$.
Set up the two integrals:

$$\int_{-1}^{0}\left(y^3 - y\right)dy + \int_{0}^{1}\left(y - y^3\right)dy$$

And integrate them:

$$\left(\frac{y^4}{4} - \frac{y^2}{2}\right)\Bigg|_{-1}^{0} + \left(\frac{y^2}{2} - \frac{y^4}{4}\right)\Bigg|_{0}^{1} = \frac{1}{2}$$

PRACTICE PROBLEM SET 25

Find the area of the region between the two curves in each problem, and be sure to sketch each one. (We only gave you endpoints in one of them.) The answers are in Chapter 21.

1. The curve $y = x^2 - 2$ and the line $y = 2$.

2. The curve $y = x^2$ and the curve $y = 4x - x^2$.

3. The curve $y = x^3$ and the curve $y = 3x^2 - 4$.

4. The curve $y = x^2 - 4x - 5$ and the curve $y = 2x - 5$.

5. The curve $y = x^3$ and the x-axis, from $x = -1$ to $x = 2$.

6. The curve $x = y^2$ and the line $x = y + 2$.

7. The curve $x = y^2$ and the curve $x = 3 - 2y^2$.

8. The curve $x = y^3 - y^2$ and the line $x = 2y$.

9. The curve $x = y^2 - 4y + 2$ and the line $x = y - 2$.

10. The curve $x = y^{\frac{2}{3}}$ and the curve $x = 2 - y^4$.

SET 25

1. $\frac{32}{3}$

2. $\frac{8}{3}$

3. $\frac{27}{4}$

4. 36

5. $\frac{17}{4}$

6. $\frac{9}{2}$

7. 4

8. $\frac{37}{12}$

9. $\frac{9}{2}$

10. $\frac{12}{5}$

PRACTICE PROBLEM SET 26

Calculate the volumes below. The answers are in Chapter 21.

1. Find the volume of the solid that results when the region bounded by $y=\sqrt{9-x^2}$ and the x-axis is revolved around the x-axis.

2. Find the volume of the solid that results when the region bounded by $y = \sec x$ and the x-axis from $x=-\frac{\pi}{4}$ to $x=\frac{\pi}{4}$ is revolved around the x-axis.

3. Find the volume of the solid that results when the region bounded by $x=1-y^2$ and the y-axis is revolved around the y-axis.

4. Find the volume of the solid that results when the region bounded by $x=\sqrt{5}y^2$ and the y-axis from $y = -1$ to $y = 1$ is revolved around the y-axis.

5. Find the volume of the solid that results when the region bounded by $y = x^3$, $x = 2$, and the x-axis is revolved around the line $x = 2$.

6. Use the method of cylindrical shells to find the volume of the solid that results when the region bounded by $y = x$, $x = 2$, and $y=-\frac{x}{2}$ is revolved around the y-axis.

7. Use the method of cylindrical shells to find the volume of the solid that results when the region bounded by $y=\sqrt{x}$, $y = 2x - 1$, and $x = 0$ is revolved around the y-axis.

8. Use the method of cylindrical shells to find the volume of the solid that results when the region bounded by $y = x^2$, $y = 4$, and $x = 0$ is revolved around the x-axis.

9. Use the method of cylindrical shells to find the volume of the solid that results when the region bounded by $y=2\sqrt{x}$, $x = 4$, and $y = 0$ is revolved around the y-axis.

10. Use the method of cylindrical shells to find the volume of the solid that results when the region bounded by $y^2 = 8x$ and $x = 2$ is revolved around the line $x = 4$.

11. Find the volume of the solid whose base is the region between the semi-circle $y=\sqrt{16-x^2}$ and the x-axis, and whose cross-sections perpendicular to the x-axis are squares with a side on the base.

12. Find the volume of the solid whose base is the region between $y = x^2$ and $y = 4$ and whose perpendicular cross-sections are isosceles right triangles with the hypotenuse on the base.

SET 26

1. 36π
2. 2π
3. $\frac{16\pi}{15}$
4. 2π
5. $\frac{16\pi}{5}$
6. 8π
7. $\frac{7\pi}{15}$
8. $\frac{128\pi}{5}$
9. $\frac{256\pi}{5}$
10. $\frac{896\pi}{15}$
11. $\frac{256}{3}$
12. $\frac{128}{15}$

PRACTICE PROBLEM SET 27

Evaluate each of the following integrals. The answers are in Chapter 21.

1. $\int x \csc^2 x \, dx$

2. $\int xe^{2x} dx$

3. $\int \frac{\ln x}{x^2} dx$

4. $\int x^2 \cos x \, dx$

5. $\int x^2 \ln x \, dx$

6. $\int x \sin 2x \, dx$

7. $\int \ln^2 x \, dx$

8. $\int x \sec^2 x \, dx$

SET 27

1. $-x\cot x + \ln|\sin x| + C$

2. $\frac{x}{2}e^{2x} - \frac{1}{4}e^{2x} + C$

3. $-\frac{1}{x}\ln x - \frac{1}{x} + C$

4. $x^2\sin x + 2x\cos x - 2\sin x + C$

5. $\frac{x^3}{3}\ln x - \frac{x^3}{9} + C$

6. $-\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x + C$

7. $x\ln^2 x - 2x\ln x + 2x + C$

8. $x \cdot \tan x + \ln|\cos x| + C$

And use u-substitution. Let $u = x - 2$ and $du = dx$:

$$\int \frac{du}{\sqrt{1-u^2}}$$

Now, this looks familiar. Once you integrate, you get:

$$\sin^{-1} u + C$$

After you substitute back it becomes:

$$\sin^{-1}(x-2) + C$$

PRACTICE PROBLEM SET 28

Here is some more practice work on derivatives and integrals of inverse trig functions. The answers are in Chapter 21.

1. Find the derivative of $\frac{1}{4}\tan^{-1}\left(\frac{x}{4}\right)$.

2. Find the derivative of $\sin^{-1}\left(\frac{1}{x}\right)$.

3. Find the derivative of $\tan^{-1}\left(e^x\right)$.

4. Evaluate $\int \frac{dx}{x\sqrt{x^2-\pi}}$.

5. Evaluate $\int \frac{dx}{7+x^2}$.

6. Evaluate $\int \frac{dx}{x\left(1+\ln^2 x\right)}$.

7. Evaluate $\int \frac{\sec^2 x\, dx}{\sqrt{1-\tan^2 x}}$.

8. Evaluate $\int \frac{dx}{\sqrt{9-4x^2}}$.

9. Evaluate $\int \frac{e^{3x}dx}{1+e^{6x}}$.

SET 28.

1. $\dfrac{1}{16+x^2}$

2. $\dfrac{-1}{|x|\sqrt{x^2-1}}$

3. $\dfrac{e^x}{1+e^{2x}}$

4. $\dfrac{1}{\sqrt{\pi}} \sec^{-1} \dfrac{x}{\sqrt{\pi}} + C$

5. $\dfrac{1}{\sqrt{7}} \tan^{-1} \dfrac{x}{\sqrt{7}} + C$

6. $\tan^{-1}(\ln x) + C$

7. $\sin^{-1}(\tan x) + C$

8. $\dfrac{1}{2} \sin^{-1} \dfrac{2x}{3} + C$

9. $\dfrac{1}{3} \tan^{-1}(e^{3x}) + C$

PRACTICE PROBLEM SET 29

Evaluate the following integrals. The answers are in Chapter 21.

1. $\int \sin^4 x\, dx$

2. $\int \cos^4 x\, dx$

3. $\int \cos^4 x \sin x\, dx$

4. $\int \sin^3 x \cos^5 x\, dx$

5. $\int \sin^2 x \cos^2 x\, dx$

6. $\int \tan^3 x \sec^2 x\, dx$

7. $\int \tan^5 x dx$

8. $\int \cot^2 x \sec x\, dx$

$\boxed{\text{SET 29.}}$

1. $\frac{3x}{8} - \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + C$

2. $\frac{3x}{8} + \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + C.$

3. $-\frac{\cos^5 x}{5} + C$

4. $-\frac{\cos^6 x}{6} + \frac{\cos^8 x}{8} + C$

5. $\frac{x}{8} - \frac{\sin 4x}{32} + C$

6. $\frac{\tan^4 x}{4} + C$

7. $\frac{\tan^4 x}{4} - \frac{\tan^2 x}{2} - \ln|\cos x| + C$

8. $-\csc x + C$

dt dt

Plug these into the formula:

$$L = \int_0^{\frac{\pi}{4}} \sqrt{\sec^4 t + \sec^2 t \tan^2 t}\, dt$$

PRACTICE PROBLEM SET 30

Find the length of the following curves between the specified intervals. Evaluate the integrals unless the directions state otherwise. The answers are in Chapter 21.

1. $y = \frac{x^3}{12} + \frac{1}{x}$ from $x = 1$ to $x = 2$
2. $y = \tan x$ from $x = -\frac{\pi}{6}$ to $x = 0$ (Set up but do not evaluate the integral.)
3. $y = \sqrt{1 - x^2}$ from $x = 0$ to $x = \frac{1}{4}$ (Set up but do not evaluate the integral.)
4. $x = \frac{y^3}{18} + \frac{3}{2y}$ from $y = 2$ to $y = 3$
5. $x = \sqrt{1 - y^2}$ from $y = -\frac{1}{2}$ to $y = \frac{1}{2}$ (Set up but do not evaluate the integral.)
6. $x = \sin y - y \cos y$ from $y = 0$ to $y = \pi$ (Set up but do not evaluate the integral.)
7. $x = \cos t$ and $y = \sin t$ from $t = \frac{\pi}{6}$ to $t = \frac{\pi}{3}$
8. $x = \sqrt{t}$ and $y = \frac{1}{t^3}$ from $t = 1$ to $t = 4$ (Set up but do not evaluate the integral.)
9. $x = 3t^2$ and $y = 2t$ from $t = 1$ to $t = 2$ (Set up but do not evaluate the integral.)

THE METHOD OF PARTIAL FRACTIONS

This is the last technique you'll learn to evaluate integrals. There are many, many more types of integrals and techniques to learn; in fact, there are courses in college primarily concerned with integrals and their uses! Fortunately for you, they're not on the AP exam (and therefore, not in this book). The BC exam usually has a partial fractions integral or two, and the concept isn't terribly hard.

We use the method of partial fractions to evaluate certain types of integrals that contain rational expressions. First, let's discuss the type of algebra you'll be doing.

If you wanted to add the expressions $\frac{3}{x-1}$ and $\frac{5}{x+2}$, you would do the following:

$$\frac{3}{x-1} + \frac{5}{x+2} = \frac{3(x+2) + 5(x-1)}{(x-1)(x+2)} = \frac{8x+1}{(x-1)(x+2)}$$

SET 30

1. $\frac{13}{12}$

2. $\int_{\frac{\pi}{6}}^{0} \sqrt{1+\sec^4 x}\, dx$

3. $\int_{0}^{\frac{1}{4}} \sqrt{\frac{1}{1-x^2}}\, dx$

4. $\frac{47}{36}$

5. $\int_{-\frac{1}{2}}^{\frac{1}{2}} \sqrt{\frac{1}{1-y^2}}\, dy$

6. $\int_{0}^{\pi} \sqrt{1+y^2\sin^2 y}\, dy$

7. $\frac{\pi}{6}$

8. $\frac{1}{2}\int_{1}^{4} \frac{\sqrt{t^7+36}}{t^4}\, dt$

9. $2\int_{1}^{2} \sqrt{9t^2+1}\, dt$

PRACTICE PROBLEM SET 31

Evaluate the following integrals. The answers are in Chapter 21.

1. $\int \frac{x+4}{(x-1)(x+6)}\,dx$

2. $\int \frac{x}{(x-3)(x+1)}\,dx$

3. $\int \frac{1}{x^3+x^2-2x}\,dx$

4. $\int \frac{2x+1}{x^2-7x+12}\,dx$

5. $\int \frac{2x-1}{(x-1)^2}\,dx$

6. $\int \frac{1}{(x+1)(x^2+1)}\,dx$

7. $\int \frac{2x+1}{(x+1)(x^2+2)}\,dx$

8. $\int \frac{x^2+3x-1}{x^3-1}\,dx$

$\boxed{\text{SET 31}}$

1. $\frac{5}{7}\ln|x-1| + \frac{2}{7}\ln|x+6| + C \quad B = \frac{1}{4}$

2. $\frac{3}{4}\ln|x-3| + \frac{1}{4}\ln|x+1| + C$

3. $-\frac{1}{2}\ln|x| + \frac{1}{6}\ln|x+2| + \frac{1}{3}\ln|x-1| + C$

4. $-7\ln|x-3| + 9\ln|x-4| + C$

5. $2\ln|x-1| - \frac{1}{x-1} + C$

6. $\frac{1}{2}\ln|x+1| - \frac{1}{4}\ln(x^2+1) + \frac{1}{2}\tan^{-1}x + C$

7. $-\frac{1}{3}\ln|x+1| + \frac{1}{6}\ln(x^2+2) + \frac{5\sqrt{2}}{6}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + C$

8. $\ln|x-1| + \frac{4}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}} + C$

PRACTICE PROBLEM SET 32

Evaluate the following integrals. The answers are in Chapter 21.

1. $\int_0^1 \frac{dx}{\sqrt{x}}$

2. $\int_{-1}^1 \frac{dx}{x^{\frac{2}{3}}}$

3. $\int_{-\infty}^0 e^x dx$

4. $\int_{-\infty}^{\frac{\pi}{2}} \tan\theta \, d\theta$

5. $\int_1^4 \frac{dx}{1-x}$

6. $\int_0^1 \frac{x+1}{\sqrt{x^2+2x}} dx$

7. $\int_{-\infty}^0 \frac{dx}{(2x-1)^3}$

8. $\int_{-\infty}^{\infty} x^3 dx$

9. $\int_0^3 \frac{dx}{x-2}$

10. $\int_{-1}^8 \frac{dx}{\sqrt[3]{x}}$

$\boxed{\text{SET 32}}$

1. 2
2. 6
3. 1
4. Diverges
5. Diverges
6. $\sqrt{3}$
7. $-\frac{1}{4}$
8. Diverges.
9. Diverges.
10. $\frac{9}{2}$

PRACTICE PROBLEM SET 33

Do the following problems on your own. The answers are in Chapter 21.

1. Find the slope of the curve $r = 2\cos 4\theta$.
2. Find the slope of the curve $r = 2 - 3\sin\theta$ at $(2, \pi)$.
3. Find the area inside the limaçon $r = 4 + 2\cos\theta$.
4. Find the area inside one loop of the lemniscate $r^2 = 4\cos 2\theta$.
5. Find the area inside $r = 2\cos\theta$ and outside $r = 1$.
6. Find the area inside the lemniscate $r^2 = 6\cos 2\theta$ and outside the circle $r = \sqrt{3}$.

SET 33

1. $$\frac{4\sin 4\theta \sin\theta - \cos 4\theta \cdot \cos\theta}{4\sin 4\theta \cdot \cos\theta + \cos 4\theta \cdot \sin\theta}$$

2. $\frac{2}{3}$

3. 18π

4. 2.

5. $\frac{\pi}{3} + \frac{\sqrt{3}}{2}$

6. $\frac{3\sqrt{3}}{2} - \frac{\pi}{2}$

PRACTICE PROBLEM SET 34

Now try these problems. The answers are in Chapter 21.

1. If $\dfrac{dy}{dx}=\dfrac{7x^2}{y^3}$ and $y(3) = 2$, find an equation for y in terms of x.

2. If $\dfrac{dy}{dx}= 5x^2 y$ and $y(0) = 6$, find an equation for y in terms of x.

3. If $\dfrac{dy}{dx}=\dfrac{1}{y+x^2y}$ and $y(0) = 2$, find an equation for y in terms of x.

4. If $\dfrac{dy}{dx}=\dfrac{e^x}{y^2}$ and $y(0) = 1$, find an equation for y in terms of x.

5. If $\dfrac{dy}{dx}=\dfrac{y^2}{x^3}$ and $y(1) = 2$, find an equation for y in terms of x.

6. If $\dfrac{dy}{dx}=\dfrac{\sin x}{\cos y}$ and $y(0) = \dfrac{3\pi}{2}$, find an equation for y in terms of x.

7. A colony of bacteria grows exponentially and the colony's population is 4,000 at time $t = 0$ and 6,500 at time $t = 3$. How big is the population at time $t = 10$?

8. A rock is thrown upward with an initial velocity, $v(t)$, of 18 m/s from a height, $h(t)$, of 45 m. If the acceleration of the rock is a constant -9 m/s^2, find the height of the rock at time $t = 4$.

9. The rate of growth of the volume of a sphere is proportional to its volume. If the volume of the sphere is initially 36π ft^3, and expands to 90π ft^3 after 1 second, find the volume of the sphere after 3 seconds.

10. A radioactive element decays exponentially in proportion to its mass. One-half of its original amount remains after 5,750 years. If 10,000 grams of the element are present initially, how much will be left after 1,000 years?

11. Use Euler's Method, with $h=0.25$, to estimate $y(1)$ if $y'=y-x$ and $y(0)=2$.

12. Use Euler's Method, with $h=0.2$, to estimate $y(1)$ if $y'=-y$ and $y(0)=1$.

13. Use Euler's Method, with $h=0.1$, to estimate $y(0.5)$ if $y'=4x^3$ and $y(0)=0$.

14. Sketch the slope field for $\dfrac{dy}{dx}=2x$.

15. Sketch the slope field for $\dfrac{dy}{dx}=-\dfrac{x}{y}$.

16. Sketch the slope field for $\dfrac{dy}{dx}=\dfrac{x}{y}$.

SET 34

1. $y = \sqrt[4]{\frac{28x^3}{3} - 236}$

2. $y = 6e^{\frac{5x^3}{3}}$

3. $y = \sqrt{2\tan^{-1}x + 4}$

4. $y = \sqrt[3]{3e^x - 2}$

5. $y = 2x^2$

6. $y = \sin^{-1}(-\cos x)$

7. 20,000 (app.)

8. 45 m

9. $\frac{1125\pi}{2} \doteq 1800 \text{ ft}^3$

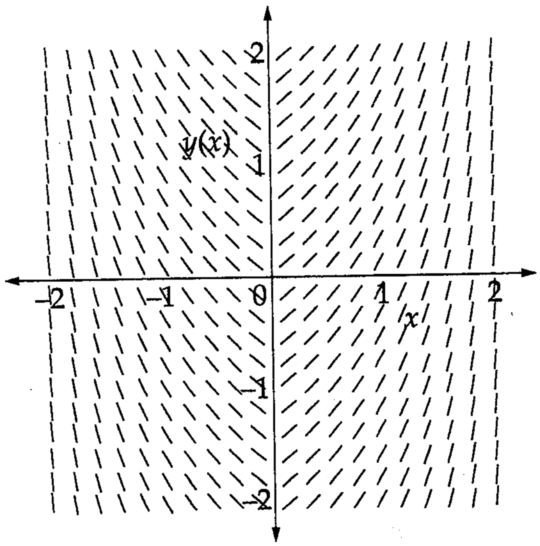
10. 8,900 grams

11. 4.441

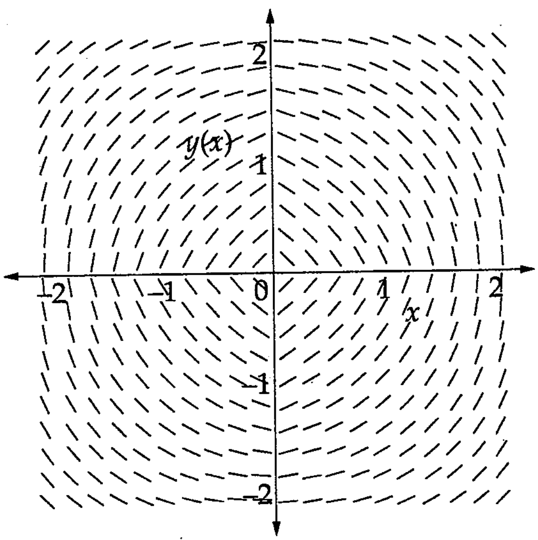
12. 0.328

13. 0.04

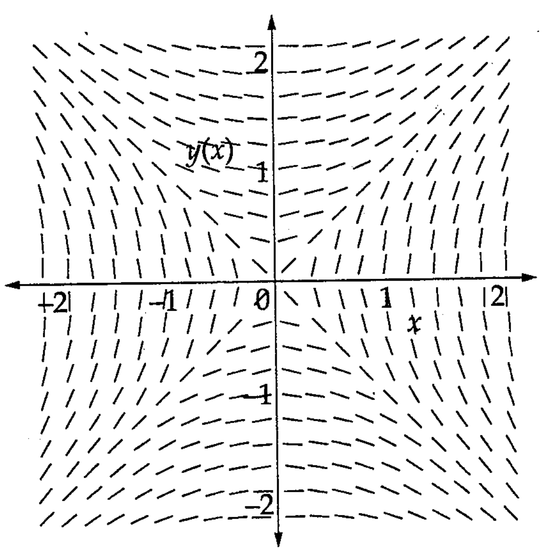
14.



15.



16.



PRACTICE PROBLEM SET 35

Now try these problems involving the series we've discussed in this chapter. The answers are in Chapter 21.

1. Find the sum of the series $2+\frac{2}{5}+\frac{2}{25}+\ldots+\frac{2}{5^4}$.

2. Find the sum of the series $8+\frac{8}{7}+\frac{8}{49}+\frac{8}{343}+\frac{8}{7^4}+\ldots$.

3. Does the series $\sum_{n=1}^{\infty}\frac{5^n}{(n-1)!}$ converge or diverge?

4. Does the series $\sum_{n=1}^{\infty}\frac{5^n}{n^2}$ converge or diverge?

5. Find the Taylor series about $a = 0$ generated by $f(x) = \cos x$.

6. Find the Taylor series about $a = 0$ generated by $f(x) = \ln(1 + x)$.

7. Find the Taylor series about $a = 0$ generated by $f(x) = e^{-x}$.

8. Find the first three nonzero terms of the Taylor series about $a=\frac{\pi}{3}$ generated by $f(x) = \sin x$.

9. Find the radius and interval of convergence for the series $\sum_{n=0}^{\infty}3^n x^n$.

10. Find the radius and interval of convergence for the series $\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}$.

11. Estimate cos (.2) using a fourth degree Taylor polynomial about $a = 0$ and find the error bound.

12. Estimate ln (1.3) using a third degree Taylor polynomial about $a = 0$ and find the error bound.

13. Determine whether $\sum_{n=2}^{\infty}\frac{1}{n\ln n}$ converges.

14. Determine whether $\sum_{n=1}^{\infty}\frac{1}{n\sqrt[4]{n}}$ converges.

15. Determine whether $\sum_{n=2}^{\infty}\frac{\ln n}{n+1}$ converges.

16. Determine whether $\sum_{n=0}^{\infty}\frac{1}{n!}$ converges.

SET 35.

1. 2.499

2. $\frac{28}{3}$

3. Converges

4. Diverges

5. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$

6. $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{k+1}}{k+1}$

7. $e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{x^k}{k!}$

8. $\frac{\sqrt{3}}{2} + \frac{1}{2}\left(x - \frac{\pi}{3}\right) - \frac{\sqrt{3}}{4}\left(x - \frac{\pi}{3}\right)^2$

9. $R = \frac{1}{3}$, Interval $\left(-\frac{1}{3}, \frac{1}{3}\right)$

10. $R = \infty$, Interval $(-\infty, \infty)$

11. 0.980067; $8.\bar{8} \times 10^{-8}$

12. 0.264 ; 0.002025

13. Diverges by Integral Test

14. Conv. by Int Test.

15. Div. by Comp. Test.

16. Cov. by Comp. Test.